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TIME SERIES IN M DIMENSIONS: SPATIAL MODELS

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ABSTRACT

The general theory of stationary spatial models is developed: namely MA, moving average; AR, autoregressive; and ARMA, autoregressive moving average processes. As compared to the time series in m dimensions, spatial models may be one-sided, two-sided, or mixed. Free use is made of the previous results of Aroian and his associates in time series in m dimensions. The main theoretical properties of the models in the univariate case are established. The multivariate case is even more important than the univariate. Estimation by minimum variance and simulation of the models are included.

1. INTRODUCTION

The results of time series in m dimensions by Aroian and his coauthors are used to establish the results of spatial models in m dimensions. If m=1 the results apply to events on a line such as a river at a particular time; for m=2, the events are those in the plane such as ecological distribution of a plant, the average rainfall for the plane after a storm is completed; for m=3, pollution is space or distribution of a mineral in a mine.

Important assumptions are outlined: the characteristic of an event in space is given by

$$z_x, x=(x_1,x_2,\ldots,x_m),\ -\infty<x<\infty,$$
$$x-\ell=(x_1-\ell_1,x_2-\ell_2,\ldots,x_m-\ell_m).$$

Weak stationarity is assumed in space as a minimum assumption:

$$\mu_z=E(z_x)=0,\quad \sigma_z^2=E(z_x-\mu_z)^2$$
$$E(a_x)=0,\ \sigma_a^2>0,\ \rho_\ell=E(z_x z_{x-\ell}-\mu_z)^2/\sigma_z^2,$$
$$\ell=(\ell_1,\ell_2,\ldots,\ell_m).$$

All second order moments exist. Note x may be in any coordinate system, and ℓ may be plus or minus; the results are in m dimensions, only the time coordinate has been dropped from consideration. Although time is a variable, it is not spatial, so new theory must be developed.

Two results, in general, follow from time series. If m=1, one-sided spatial models are covered by Box and Jenkins (1976) if the variable t in their models is replaced by χ. Isotropic models in space m=2, where χ is the radius of a circle, are models of m=1, time series in m variables, and are discussed briefly in a later section.

2. MA, AR, AND ARMA MODELS

The two-sided theoretical spatial MA model is defined:

$$z_x=\sum_{n=-\infty}^{\infty}\psi_n a_{x-n},\ -\infty<z<\infty,\ \psi_0=1,$$
$$n=(n_1,n_2,\ldots,n_m),\ \sum_{n=-\infty}^{\infty}=\sum_{n_m=-\infty}^{\infty}\cdots\sum_{n_1=-\infty}^{\infty}, \tag{2.1}$$

a_x is an i.i.d. variable with $\mu_a=0$, $\sigma_a^2>0$, independent of z_x unless z_x or $z_{x-\ell}$ involves a_x, or $a_{x-\ell}$, and $Ea_x z_{x-\ell}=0$, unless $\ell=0$.

More usually n is finite:

$$z_x=\sum_{n=-\infty}^{q}\psi_n a_{x+n},\ \psi_0=1 \tag{2.2}$$

an MA model of spatial order p_i+q_i in each spatial variable x_i, $1\le i\le m$. If $-p_i\le n\le q_i$ the spatial model is two-sided in m; if $-p_i\le n\le -q_i$ or $0\le n\le q_i$ for all i it is one-sided in m. A model may be two-sided for certain x_i and one-sided for other x_i; such a case is called a mixed model.

As examples: for m=1,

$$z_x=\psi_1 a_{x-1}+\psi_{-1}a_{x+1}+\psi_2 a_{x-2}+\psi_{-2}a_{x+2}+a_x, \tag{2.3}$$

m=1, two-sided of order two in each variable. If $\psi_{-1}=\psi_{-2}=0$, the model is one-sided. If $\psi_{-1}=0$, $\psi_2\ne0$, $\psi_{-2}\ne0$, it is mixed.

For m=2:

$$z_{x_1,x_2}=\psi_{01}a_{x_1,x_2-1}+\psi_{10}a_{x_1-1,x_2}+\psi_{0-1}a_{x_1,x_2+1}$$
$$+\psi_{-10}a_{x_1+1,x_2}+a_{x_1,x_2} \tag{2.4}$$

of spatial order two in each variable, a two-sided model; one-sided model if $\psi_{0-1}=\psi_{-10}=0$; and mixed if $\psi_{0-1}=0$, $\psi_{-1,0}\ne0$.

The two-sided theoretical spatial AR model is defined:

$$z_x=\sum_{n=-\infty}^{\infty}\phi_n z_{x+n}+a_x,\ \phi_0=0,\ -\infty<z<\infty\ . \tag{2.5}$$

Usually n is finite $-p\le n\le q$, of spatial order p_i+q_i in each variable x_i, $1\le i\le m$. It may be two-sided, one-sided or mixed as in the MA model.

$$z_x=\phi_1 z_{x-1}+\phi_{-1}z_{x+1}+\phi_2 z_{x-2}+\phi_{-2}z_{x+2}+a_x \tag{2.6}$$

$$z_{x_1,x_2}=\phi_{01}z_{x_1,x_2-1}+\phi_{10}z_{x_1-1,x_2}+\phi_{0-1}z_{x_1,x_2+1}$$
$$+\phi_{-10}z_{x_1+1,x_2}+a_{x_1,x_2}, \tag{2.7}$$

The theoretical ARMA model for time series in m dimensions is, Voss et al (1980):

$$z_{x,t}=\sum_{n=-p}^{q}\sum_{k=1}^{r}\phi_{n,k}z_{x+n,t-k}-\sum_{n=-u}^{v}\sum_{k=1}^{s}\theta_{n,k}$$
$$a_{x+n,t-k}+a_{x,t},\ \phi_{00}=0,\ \theta_{00}=0 \tag{2.8}$$

(r,s) in the temporal domain, q+p and u+v in each spatial variable. The general case would be $-\infty<n<\infty$, $-\infty<t<\infty$. The corresponding two-sided ARMA spatial model is

$$z_x=\sum_{n=-p}^{q}\phi_n z_{x+n}-\sum_{n=-u}^{v}\theta_n a_{x+n}+a_x,\ \phi_0=\theta_0=0. \tag{2.9}$$

Examples for m=1 and 2 respectively are:

$$z_x=\phi_1 z_{x-1}+\phi_2 z_{x-2}+\phi_{-1}z_{x+1}+\phi_{-2}z_{x+2}-\phi_1 z_{x-1}$$
$$-\theta_1 a_{x-1}-\theta_2 a_{x-2}-\theta_{-1}a_{x+1}-\theta_{-2}a_{x+2}+a_x \tag{2.10}$$

ARMA model two-sided p=q=2, m=1.

$$z_{x_1,x_2}=\phi_{01}z_{x_1,x_2-1}+\phi_{10}z_{x_1-1,x_2}+\phi_{0-1}z_{x_1,x_2+1}$$
$$+\phi_{-10}z_{x_1+1,x_2}-\theta_{01}a_{x_1,x_2-1}-\theta_{10}a_{x_1-1,x_2} \tag{2.11}$$

$$-\theta_{0-1}a_{x_1,x_2+1}-\theta_{-10}a_{x_1+1,x_2}+a_{x_1,x_2} \quad (2.11)$$

ARMA model, two-sided $p=q=1$, $m=2$.

The corresponding two-sided MA (AR) model is found by setting $\phi_{\pm1}=\phi_{\pm2}=\phi_{\pm3}=0$ $(\theta_{\pm1}=\theta_{\pm2}=\theta_{\pm3}=0)$ and the one-sided ARMA model by setting

$\phi_i=0$, or $\phi_{-i}=0$ and $\theta_i=0$ or $\theta_{-i}=0$.

There may be a mixture such as an ARMA model with one-sided AR or MA while the other part is two-sided.

3. MA MODELS

The simplest MA model

$$z_x=-\theta_1 a_{x-1}-\theta_{-1}a_{x+1}+a_x \quad (3.1)$$

is obtained from (2.3) by replacing ψ's by $-\theta$'s and setting $\psi_{-2}=-\theta_{-2}$, $\psi_2=-\theta_2$, and $\theta_2=\theta_{-2}=0$, $m=1$. For this model

$$\sigma_z^2=\sigma_a^2(1+\theta_1^2+\theta_{-1}^2) \quad (3.2)$$

$$\rho_1=-(\theta_1+\theta_{-1})/(1+\theta_1^2+\theta_{-1}^2),$$

$$\rho_2=(\theta_1\theta_{-1})/(1+\theta_1^2+\theta_{-1}^2),\ \rho_1=\rho_{-1},\ \rho_2=\rho_{-2},$$

and all other $\rho_\ell=0$, $\ell>2$.

If $\theta_{-1}=0$, $\theta_1\neq0$, (3.1) reduces to the *one-sided* model in t; and also if $\theta_1=0$, $\theta_{-1}\neq0$. Given $\{\theta_1,\ \theta_2\}$, $\{\rho_1,\rho_2\}$ are given by (3.2). Conversely given $(\hat\rho_1,\hat\rho_2)$ a sample estimate of (ρ_1,ρ_2), $(\hat\theta_1,\hat\theta_2)$ may be found from the set replacing population values by estimates:

$$1+\theta_1^2+\theta_{-1}^2=-(\theta_1+\theta_{-1})/\rho_1=\theta_1\theta_{-1}/\rho_2;\ \text{and}$$

$$\rho_1=-(1/\theta_1+1/\theta_{-1})\rho_2, \quad (3.3)$$

Set $\theta_1=u+v$, $\theta_{-1}=u-v$, then

$$1+2(u^2+v^2)=-2u/\rho_1=(u^2-v^2)/\rho_2, \quad (3.4)$$

whose solution involves the intersection of a circle and hyperbola. This may lead to four possible sets, but the condition

$$|\theta_1|+|\theta_{-1}|<1 \quad (3.5)$$

limits the results to one set.

If $\theta_1=\theta_{-1}$, $\rho_1=0$, and if θ_1 or $\theta_{-1}=0$, $\rho_2=0$. Table 1 lists the values of $\{\rho_1,\rho_2\}$ given $\{\theta_1,\theta_2\}$, and conversely. The manipulations of (3.4) show

$$|\rho_2|\le 1/3, \text{ and } (1+2\rho_2)^2\ge 4\rho_1^2.$$

Only the values of $\{\rho_1,\rho_2\}$ for $-1\le\theta_1\le1$, and $\theta_{-1}<0$ are tabled, since the remaining values of $\{\theta_1,\theta_2\}$ may be found from the skew symmetry implied by (3.3).

The characteristic equation of (3.1) is $1-\theta_1 B_x-\theta_{-1}B_x^{-1}$, and the corresponding AR representation of the MA model is:

$$a_x=\sum_{d=0}^{\infty}(\theta_1 B_x+\theta_{-1}B_x^{-1})^d z_x,\ a_x=z_x+\theta_1 z_{x-1}$$
$$+\theta_{-1}z_{x+1}+\theta_1^2 z_{x-2}+2\theta_1\theta_{-1}z_x+\theta_{-1}^2 z_{x-2}+\dots$$

$$\text{or}\quad a_x=\pi_0 z_x+\pi_1 z_{x-1}+\pi_{-1}z_{x+1}+\dots \quad (3.6)$$

Note that $\pi_0\neq1$, and π_n exists and the series converges for the values of $|\theta_1|+|\theta_{-1}|<1$.

Given a sample of n, $\{\hat\theta_1,\hat\theta_{-1}\}$ is estimated from (3.3) using sample estimates $\{\hat\rho_1,\hat\rho_2\}$ and the methods of solution already indicated. The approximate variance of $\hat\theta$ is $\hat\sigma_{\hat\theta}^2=(1-\hat\theta^2)/n$, and confidence intervals may be obtained if it is assumed that the errors are distributed normally. Another method of estimation is to vary θ, and choose the θ which minimizes the variance of the error of prediction, $e=z_x-\hat z_x$.

Another simple MA model m=2, of the first order is:

$$z_{x_1,x_2}=-\theta_{01}a_{x_1,x_2-1}-\theta_{10}a_{x_1-1,x_2}-\theta_{0-1}a_{x_1,x_2+1}$$
$$-\theta_{-10}a_{x_1+1,x_2}+a_{x_1,x_2} \quad (3.7)$$

The characteristic function is:

$$1-\theta_{01}B_{x_2}-\theta_{10}B_{x_1}-\theta_{0-1}B_{x_2}^{-1}-\theta_{-10}B_{x_1}^{-1}, \quad (3.8)$$

the values of σ_z^2, $\rho_{01},\rho_{10},\rho_{02},\rho_{20},\rho_{11},\rho_{-11}$ are:

$$\sigma_z^2=\sigma_a^2(1+\theta_{01}^2+\theta_{0-1}^2+\theta_{10}^2+\theta_{-10}^2),$$

$$\rho_{01}=\rho_{0-1}=-(\theta_{01}+\theta_{0-1})/\sigma_z^2\sigma_a^{-2},$$

$$\rho_{10}=\rho_{-10}=-(\theta_{10}+\theta_{-10})/\sigma_z^2\sigma_a^{-2},$$

$$\rho_{02}=\rho_{0-2}=(\theta_{01}\theta_{0-1})/\sigma_z^2\sigma_a^{-2}$$

$$\rho_{20}=\rho_{-20}=(\theta_{10}\theta_{-10})/\sigma_z^2\sigma_a^{-2}$$

$$\rho_{11}=\rho_{-1-1}=(\theta_{01}\theta_{10}+\theta_{0-1}\theta_{-10})/\sigma_z^2\sigma_a^{-2}$$

$$\rho_{-11}=\rho_{1-1}=(\theta_{01}\theta_{-10}+\theta_{0-1}\theta_{10})\sigma_z^2\sigma_a^{-2} \quad (3.9)$$

$$1+\theta_{01}^2+\theta_{0-1}^2+\theta_{10}^2+\theta_{-10}^2=-(\theta_{01}+\theta_{0-1})\rho_{01}^{-1}=-(\theta_{10}+\theta_{-10})$$

$$\rho_{10}^{-1}=(\theta_{01}\theta_{10}+\theta_{0-1}\theta_{-10})\rho_{11}^{-1}=(\theta_{01}\theta_{-10}+\theta_{0-1}\theta_{10})$$

$$\rho_{-11}^{-1}=(\theta_{01}\theta_{0-1})\rho_{02}^{-1}=(\theta_{10}\theta_{-10})\rho_{20}^{-1}. \quad (3.10)$$

The corresponding AR model is given by the inversion of the characteristic function in the usual way. Note $|\theta_{01}|+|\theta_{10}|+|\theta_{0-1}|+|\theta_{-10}|<1$. Given a set $(\hat\rho_{01},\hat\rho_{10},\hat\rho_{11},\hat\rho_{-1,1})$ the proper set of equations from (3.10) are used to estimate $(\hat\theta_{01},\hat\theta_{10},\hat\theta_{0-1},\hat\theta_{-1,0})$. The approximate variances of the set of the θ's may be found using the methods suggested in Aroian and Taneja (1980), Perry and Aroian (1979), and Aroian and Schmee (1980). The variance of the error of predictions may be minimized by changing the vector of estimates θ until a minimum is found for this variance, see Aroian and Taneja (1980).

Simulation of MA(1,1) Model

Let $\theta_1=.2$, $\theta_{-1}=-.2$ in (3.1), a_x's being distributed as N(0,1). This model is simulated with 100 observations. First random shocks a_x's are

generated, and z_x's are found from (3.1). The estimated correlation coefficients r_1, r_2 are obtained. The $\{\hat{\theta}_1, \hat{\theta}_{-1}\}$ is found from (3.3)-(3.4) using r_1, r_2. Minimum error prediction of θ_1 and θ_{-1} are also found by using a_x's obtained from z_x's by minimizing $e_x=(z-z_x)$, this estimate is $\{\hat{\theta}_1, \hat{\theta}_{-1}\}=\{.330, -.340\}$, note, that given $z_x, a_x = z_x+\hat{\theta}_1 a_{x-1}+\hat{\theta}_{-1}a_{x+1}$. With twenty-five simulation runs it is found that $\bar{\theta}_1=.284$, $\bar{\theta}_{-1}=-.239$, while minimum variance estimates are $\bar{\theta}_1=.285$, $\bar{\theta}_{-1}=-.237$. From $\hat{\sigma}_\theta^2 \sim (1-\hat{\theta}^2)/n$, $\hat{\sigma}_{\theta_1}^2=\hat{\sigma}_{\theta-1}^2=.0384$. The approximate formula for the covariance $\gamma(\hat{\theta}_1, \hat{\theta}_{-1}) \sim -\theta_1(1+\theta_{-1})n^{-1}$ or $-.00784$ for the simulated case versus the actual of $-.0032$.

4. AR MODELS

Some simple AR models m=1 and m=2 are analyzed to show how results may be obtained. Note that in all fully two-sided AR models $\phi_{-i} = \phi_i$.

The two-sided simplest AR model, m=1, from (2.6) is

$$z_x = \phi_1 z_{x-1}+\phi_{-1}z_{x+1}+a_x=\phi_1(a_{x-1}+z_{x+1})+a_x,$$

$$\sigma_z^2=(1-2\rho_1\phi_1)^{-1}\ \sigma_a^2 . \tag{4.1}$$

Since

$\rho_\ell=\phi_1(\rho_{\ell-1}+\rho_{\ell+1})$ for all ℓ, $\ell\neq 0$, then

$$\rho_\ell=\rho_1^\ell,\ \phi_1=\rho_1/(1+\rho_2)=\rho_1/(1+\rho_1^2), \text{ a parabola.} \tag{4.2}$$

Note $|\phi_1|<\tfrac{1}{2}$, and given ϕ_1, $\rho_1=0.5\ \phi_1^{-1}\{1\pm(1-4\phi_1^2)^{\frac{1}{2}}\}$. The values of $\{\phi_1, \rho_1, \rho_2, \rho_3\}$ are given in Table 2.

The AR model written as an MA model is:

$$z_x= \{\sum_{i=0}^{\infty} \phi_1^i(B_x+B_x^{-1})^i\}a_x=a_x+\phi_1(a_{x+1}+a_{x-1})$$

$$+\ \phi_1^2(a_{x+2}+2a_x+a_{x-2}) + \dots;$$

if $\quad z_x= \sum_{-\infty}^{\infty} \pi_{-i}\ a_{x+i}$, $\pi_0=1+\phi_1(\pi_1+\pi_{-1})$,

$$\pi_1= \phi_1(\pi_0+\pi_2),\ \pi_{-1}=\phi_1(\pi_{-2}+\pi_0)\ \dots,$$

$$\pi_i= \phi_1(\pi_{i-1}+\pi_{i+1}),\ \pi_{-1}=\phi_1(\pi_{-i-1}+\pi_{-i+1});$$

$$\text{and} \quad \pi_{-i}=\pi_i . \tag{4.3}$$

Given ϕ_1, $\pi_0 = \sum_{i=0}^{\infty} \binom{2i}{i} \phi_1^{2i}$,

$$\pi_1 =\sum_{i=1}^{\infty} \binom{2i-1}{i-1} \phi_1^{2i-1},\quad \pi_2=\sum_{i=1}^{\infty} \binom{2i}{i-1} \phi_1^{2i},$$

$$\pi_3= \sum_{2}^{\infty} \binom{2i-1}{i-2} \phi_1^{2i-1},\quad \pi_4=\sum_{i=2}^{\infty} \binom{2i}{i-2} \phi_1^{2i}, \text{ etc.}$$

For $\phi_1 = .2$

$\pi_1 = .22772201$, consequently $\pi_0 = 1.09108881$, and $\pi_1/\pi_0 = .20871 = \rho_1$, which checks Table 2.

The autoregressive model given in (4.1) is simulated given that $\theta_1 = .2$, and a_x's are N(0.1). The number of observations in each simulation is 100. First a_x's are generated and z_x's are obtained by using (4.1); to do this consecutive forward and backward substitutions are performed until convergence is assured. Results of the twenty-five simulation runs are; $\bar{r}_1 = .2282$, $\hat{\sigma}_{r_1}^2 = 0.0143$, $\bar{\theta}_1 = .2091$, $\hat{\sigma}_{\theta_1}^2 = .0099$. For one particular run when $\theta_1 = .2$, theoretically $\rho_1 = .2087$ and $\sigma_{\theta_1}^2 \sim n^{-1}(1-\theta_1^2)/(1-\rho_1^2)=(1-.04)/(100(1-.0427)) = .01$. Simulated case gives $r_1 = .1556$, $\hat{\theta}_1=.1516$, $\sigma_{\theta_1}^2 \sim n^{-1}(1-\hat{\theta}^2)/(1-r_1^2) = (1-.0231)/(100(1-.0242)) = .01$. Since $\hat{\phi}_1$ using $\hat{\phi}_1=\hat{\rho}_1(1+\hat{\rho}_2)^{-1}$ is a least squares estimate so $\hat{\sigma}_\phi^2 2 = (1-\phi_1^2)/n(1-\rho_1^2)$, then confidence intervals may be found assuming the a's are distributed normally. The minimum variance estimate of $\hat{\phi}_1$ may be obtained as indicated in the case of θ's.

Another simple two-sided AR model, m=1, is:

$$z_x = \phi_1(z_{x-1}+z_{x+1}) + \phi_2(z_{x-2}+z_{x+2}) + a_x .$$

$$\sigma_z^2 = \sigma_a^2(1-2\rho_1\phi_1-2\rho_2\phi_2)^{-1} ,$$

$$\rho_\ell = \phi_1(\rho_{\ell-1}+\rho_{\ell+1}) + \phi_2(\rho_{\ell-2}+\rho_{\ell+2}), \ell\neq 0, \tag{4.4}$$

the Yule-Walker equations are:

$$\rho_1 = \phi_1(1+\rho_2) + \phi_2(\rho_1+\rho_3)$$

$$\rho_2 = \phi_1(\rho_1+\rho_3) + \phi_2(1+\rho_4) \tag{4.5}$$

$$\phi_1 =[\rho_1(1+\rho_4) - \rho_2(\rho_1+\rho_3)]/[(1+\rho_2)(1+\rho_4)-(\rho_1+\rho_3)^2]$$

$$\phi_2 =[\rho_2(1+\rho_2)-\rho_1(\rho_1+\rho_3)]/[(1+\rho_2)(1+\rho_4)-(\rho_1+\rho_3)^2] \tag{4.6}$$

$|\phi_1| + |\phi_2|<\tfrac{1}{2}$, and the equivalent MA model is

$$\sum_{i=0}^{\infty} \{\phi(B_x+B_x^{-1}) + \phi_2(B_x^2+B_x^{-2})\}^i\ a_x . \tag{4.7}$$

Given a permissible set $\{\rho_1, \rho_2, \rho_3, \rho_4\}$, $\{\phi_1, \phi_2\}$ is determined. Given $\{\phi_1, \phi_2\}$, then $\{\rho_1, \rho_2, \rho_3, \rho_4\}$ is determined from the corresponding MA model, Aroian and Schmee (1980). The variances and the covariance of $\{\phi_1, \phi_2\}$ may be found as indicated in Aroian and Schmee (1980), and as usual the minimum variance estimates of $\{\hat{\phi}_1, \hat{\phi}_2\}$. The simulation and prediction of such a model is essentially the same as given in Box and Jenkins (1976) for pure time series, but four starting values will be needed instead of only two.

For m=2, a simple two-sided model is:

$$z_{x_1,x_2} = \phi_{11}z_{x_1-1,x_2-1}+\phi_{-1-1}z_{x_1+1,x_2+1}$$

$$+\phi_{1-1}z_{x_1-1,x_2+1}+\phi_{-11}z_{x_1+1,x_2-1}+a_{x_1,x_2} \tag{4.8}$$

with $\phi_{11} = \phi_{-1-1} = \phi_1$ and $\phi_{1-1} = \phi_{-1,1} = \phi_2$;

hence
$$z_{x_1,x_2} = \phi_1(z_{x_1-1,x_2-1} + z_{x_1+1,x_2+1}) + \phi_2(z_{x_1-1,x_2+1} + z_{x_1+1,x_2-1}) + a_{x_1,x_2}$$

$$\sigma_z^2 = (1-2\rho_{11}\phi_1 - 2\rho_{-11}\phi_2)^{-1}\,\sigma_a^2, \quad (4.9)$$

$$\rho_{\ell_1,\ell_2} = \phi_1(\rho_{\ell_1-1,\ell_2-1} + \rho_{\ell_1-1,\ell_2+1}) + \phi_2(\rho_{\ell_1-1,\ell_2+1} + \rho_{\ell_1-1,\ell_2+1}),\ \ell_1 \neq 0, \ell_2 \neq 0.$$

The characteristic function is:

$$1-\phi_1(B_{x_1}B_{x_2} + B_{x_1}^{-1}B_{x_2}^{-1}) - \phi_2(B_{x_1}B_{x_2}^{-1} + B_{x_1}^{-1}B_{x_2}) \quad (4.10)$$

which may be used to find the corresponding MA model useful to obtain all needed ρ_{ℓ_1,ℓ_2}, see Aroian and Schmee (1980). Note $|\phi_1|+|\phi_2|<½$.

The Yule-Walker equations are:

$$\rho_{11} = \phi_1(1+\rho_{22}) + \phi_2(\rho_{02}+\rho_{20})$$

$$\rho_{1-1} = \phi_1(\rho_{02}+\rho_{20}) + \phi_2(1+\rho_{-22}), \text{ and} \quad (4.11)$$

$$\phi_1 = [\rho_{1-1}(1+\rho_{-22}) - \rho_{1-1}(\rho_{02}+\rho_{20})]/[(1+\rho_{22})^2 - (\rho_{02}+\rho_{20})^2]$$

$$\phi_2 = [\rho_{1-1}(1+\rho_{22}) - \rho_{11}(\rho_{02}+\rho_{20})]/[(1+\rho_{22})^2 - (\rho_{02}+\rho_{20})^2] \quad (4.12)$$

which involve $\rho_{11}, \rho_{1-1}, \rho_{02}, \rho_{20}, \rho_{2,2}$, and $\rho_{-2,2}$. *Given estimates of these six correlations* $(\hat{\phi}_1, \hat{\phi}_2)$ are found from (4.12).

Conversely, given $\{\phi_1,\phi_2\}$, then all ρ_{ℓ_1,ℓ_2} must be found from the corresponding MA expansion. Estimation, variances and covariance of (ϕ_1,ϕ_2) may be found as already indicated in other AR models, particularly the minimum variance method.

Another obvious two-sided model is:

$$z_{x_1,x_2} = \phi_1(z_{x_1,x_2-1} + z_{x_1,x_2+1}) + \phi_2(z_{x_1-1,x_2} + z_{x_1+1,x_2}) + a_{x_1,x_2} \quad (4.13)$$

$$\sigma_z^2 = (1-2\rho_{01}\phi_1 - 2\rho_{10}\phi_2)^{-1}\,\sigma_a^2\,,$$

$$\rho_{\ell_1,\ell_2} = \phi_1(\rho_{\ell_1,\ell_2-1} + \rho_{\ell_1,\ell_2+1}) + \phi_2(\rho_{\ell_1-1,\ell_2} + \rho_{\ell_1+1,\ell_2})$$
$$\ell_1 \neq 0, \ell_2 \neq 0.$$

This is analyzed in exactly the same way as the previous models.

Clearly, $|\phi_1|+|\phi_2|<½$, the characteristic function is

$$1 - \phi_1(B_{x_2} + B_{x_2}^{-1}) - \phi_2(B_{x_1} + B_{x_1}^{-1})\ , \quad (4.14)$$

from which the corresponding MA may be found and all ρ_{ℓ_1,ℓ_2}'s. The Yule-Walker equations are:

$$\rho_{01} = \phi_1(1+\rho_{02}) + \phi_2(\rho_{1-1}+\rho_{-1-1})$$

$$\rho_{10} = \phi_1(\rho_{1-1}+\rho_{-1-1}) + \phi_2(1+\rho_{20}), \quad (4.15)$$

with solution:

$$\phi_1 = [\rho_{01}(1+\rho_{20}) - \rho_{10}(\rho_{1-1}+\rho_{-1-1})]/[(1+\rho_{02})(1+\rho_{20}) - (\rho_{1-1}+\rho_{-1-1})^2]\ ,$$

$$\phi_2 = [\rho_{10}(1+\rho_{02}) - \rho_{10}(\rho_{1-1}+\rho_{-1-1})]/[(1+\rho_{02})(1+\rho_{20}) - (\rho_{1-1}+\rho_{-1-1})^2]\ . \quad (4.16)$$

Estimation, the approximate variance-covariance matrix, and confidence limits as well as minimum variance estimates are found as before.

It should be mentioned that the partial auto-correlation function of the AR models have a cut-off property, for the first model m=1, all ϕ_i, i>1 are zero, and ϕ_1 is a partial coefficient of correlation. Where there are two ϕ's, ϕ_1 and ϕ_2 both are partial coefficients of correlation, and ϕ_i, i > 2 are zero. This property is helpful in determining how far to proceed in the ϕ's; alternatives are the analysis of variance methods, and that in which the variance of the error of prediction falls as the ϕ's increase. Other possible alternatives still unexplored are the χ^2 test and the Kolmogoroff-Smirnov test. An important point to remember is, as the number of ϕ's increase, so does the variance of the forecast errors.

5. ARMA MODELS

The two models considered are simplifications of (2.10) and (2.11):

$$z_x = \phi_1(z_{x-1}+z_{x+1}) - \theta_1(a_{x-1}+a_{x+1}) + a_x, \quad (5.1)$$

$$z_{x_1,x_2} = \phi_{01}(z_{x_1,x_2-1} + z_{x_1,x_2+1}) - \theta_{01}(a_{x_1,x_2-1} + a_{x_1,x_2+1}) + a_{x_1,x_2} \quad (5.2)$$

For (5.1) the equations for σ_z^2, ρ_1, and ρ_2 are:

$$\sigma_z^2 = \sigma_a^2(1+2\theta_1^2 - 4\theta_1\phi_1)(1-2\rho_1\phi_1)-1 \quad (5.3)$$

$$\rho_1 = \phi_1(1+\rho_2) + \sigma_a^2/\sigma_z^2\ \{\phi_1\theta_1^2 - 2\theta_1\} \quad (5.4)$$

$$\rho_2 = \phi_1(\rho_1+\rho_3) + (\theta_1^2 - 2\phi_1\theta_1)\sigma_a^2/\sigma_z^2 \quad (5.5)$$

From these:

$$\sigma_z^2/\sigma_a^2 = [\rho_2 - \phi_1(\rho_1+\rho_3)]/[\theta_1^2 - 2\theta_1\phi_1] = [\rho_1 - \phi_1(1+\rho_2)]/[\{\phi_1\theta_1^2 - 2\theta_1\}] = [1-2\rho_1\phi_1]/[1+2\theta_1^2 - 4\theta_1\phi_1]. \quad (5.6)$$

Thus (5.6) may be solved for $\{\phi_1,\theta_1\}$ using $\{\rho_1,\rho_1\}$. The restrictions on $\{\phi_1,\phi_1\}$ and $\{\phi_{01},\theta_{01}\}$ for the AR and MA apply. Both (5.1) and (5.2) may be written in the equivalent AR or MA model since:

$$\{1-\phi_1(B_x+B_x^{-1})\}z_x = \{1-\theta_1(B_x+B_x^{-1})\}\,a_x,$$

$$a_x = \{\sum_{i=0}^{\infty} \theta_1^i (B_x+B_x^{-1})^i\}\{1-\theta_1(B_x+B_x^{-1})\}\ z_x\ ,$$

and

$$z_x = \{\sum_{i=0}^{\infty} \phi_1^i (B_x+B_x^{-1})^i\}\{1-\phi_1(B_x+B_x^{-1})\}\ z_x; \quad (5.7)$$

with similar results for (5.2).

A more general model than (5.1) is:

$$z_x = \phi_1(z_{x-1}+z_{x+1}) - \theta_1 a_{x-1} - \theta_{-1} a_{x+1} + a_x, \qquad (5.8)$$

which reduces to (5.1) if $\theta_{-1} = \theta_1$.

Now

$$\sigma_z^2 = \sigma_a^2(1+\theta_1^2+\theta_{-1}^2-2\phi_1\theta_1-2\phi_1\theta_{-1})(1-2\rho_1\phi_1)^{-1},$$

$$\rho_1 = \phi_1(1+\rho_2)+[\sigma_a^2/\sigma_z^2]\{\phi_1\theta_1\theta_{-1}-\theta_1-\theta_{-1}\},$$

$$\rho_2 = \phi_1(\rho_1+\rho_3)+[\sigma_a^2/\sigma_z^2]\{-\phi_1(\theta_1+\theta_{-1})+\theta_1\theta_{-1}\},$$

$$\rho_3 = \phi_1(\rho_2+\rho_4)+[\sigma_a^2/\sigma_z^2](\phi_1\theta_1\theta_{-1}) \quad . \qquad (5.9)$$

Solve (5.9) using σ_a^2/σ_z^2 from the first equation, substitute this into the other three and solve for $\{\phi_1,\theta_1,\theta_{-1}\}$ using the sample values $\{r_1,r_2,r_3,r_4\}$ for the ρ's. Write (5.8) as

$$\{1-\phi_1(B_x+B_x^{-1})\}\ z_x = \{1-\theta_1\ B_x-\theta_{-1}\ B_x^{-1}\}\ a_x\ , \qquad (5.10)$$

then z_x as an MA process is:

$$z_x = \{1-\theta_1 B_x-\theta_{-1}B_x^{-1}\}\{1-\phi_1(B_x+B_x^{-1})\}^{-1}\ a_x$$

$$z_x = [\{1-\theta_1 B_x-\theta_{-1}B_x^{-1}\}\sum_{i=0}^{\infty} \phi_1^i(B_x+B_x^{-1})^i]\ a_x\ , \qquad (5.11)$$

$|\theta_1|+|\theta_{-1}|<1$, $|\phi_1|<½$, the same conditions are in (3.5) and (4.1).

Represent z_x as an AR model from (5.10)

$$a_x = \{1-\phi_1(B_x+B_x^{-1})\}\{1-\theta_1 B_x-\theta_{-1}B_x^{-1}\}^{-1}\ z_x$$

$$a_x = [\{1-\phi_1(B_x+B_x^{-1})\}\ \sum_{i=0}^{\infty}\{\theta_1 B_x+\theta_{-1}B_x^{-1}\}^i]z_x \qquad (5.12)$$

with the same restrictions on $\{\phi_1,\theta_1,\theta_{-1}\}$ as in (5.11).

Now suppose $\phi_1,\theta_1,\theta_{-1}$ are given satisfying the restrictions in (5.11), what are the values of ρ_ℓ, the autocorrelation function?

Estimation proceeds as indicated in Aroian and Taneja (1980), by changing an ARMA model to an equivalent AR model and using the results from least squares.

6. EXAMPLES

Some examples will be completed, particularly Whittle's example of wheat data and possibly some others. Simulations will be done in a separate study as well as further extensions of these models. One sided and mixed models will be done in the future.

7. ISOTROPIC PROCESSES

Let the variable χ represent the distance from any point (χ_1,χ_2) in the plane, or the point (χ_1,χ_2,χ_3) in space. Then for any stationary isotropic process the results from Box and Jenkins (1976) may be used in all cases for MA, AR, ARMA replacing t by χ. This applies not only to stationary processes but to nonstationary processes if one uses differences as indicated there. Since the method is strightforward, no further discussion is needed. For isotropic processes in time and space, in place of $z_{x,t}$ as given in Aroian et al, one would replace x by χ, and retain t and use the methods indicated there for stationary processes. For nonstationary processes differences in two variables may be used or transformations. Another alternative to transformations or to differencing direct treatment of nonstationarity is feasible and will be investigated subsequently.

8. CONCLUSIONS

The methods of time series in m dimensions are applied to two-sided spatial models in one and two dimensions: MA, AR, and ARMA models illustrate the techniques including estimation. These results presented in this paper are based on the second order moments, and MA, AR, and ARMA models as developed in time series in m dimensions. Papers in bibliography numbered as 1,2,3,4,11,12, 14, and 15 reflect the Aroian point of view. Some other points of view related to these results may be found in Bartlett (1975), Bennett (1979), Besag (1972), Cliff and Ord (1973) and Ord (1975). Bartlett reflects a position from partial differential equations, and power spectrums to AR models, a broad point of view covering briefly most of the previous work before 1975. Bennett covers the ideas quite thoroughly and presents a comprehensive bibliography, but does not give enough details as Box and Jenkins (1976) do in their work. Ord considers only first order autoregressive models, m=1, which are restricted and not general.

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TABLE 1

Values of (ρ_1, ρ_2) and (θ_1, θ_{-1})

θ_{-1} \ θ_1	-1	-.9	-.8	-.6	-.4	-.2	.0	.2	.4	.6	.8	.9	1.0
1.	0 -.333	-.036 -.320	-.076 -.303	-.169 -.254	-.278 -.185	-.392 -.098	-.500 0	-.588 .098	-.648 .185	-.678 .227	-.682 .303	-.676 .320	-.667 .333
.9	.036 -.320	0 -.309	-.082 -.294	-.138 -.249	-.254 -.183	-.379 -.097	-.497 0	-.595 .097	-.660 .183	-.691 .249	-.694 .294	-.687 .309	-.676 .320
.8	.076 -.303	.082 -.294	0 -.211	-.100 -.240	-.222 -.178	-.357 -.095	-.487 0	-.595 .095	-.667 .178	-.700 .240	-.701 .211	-.694 .294	-.682 .303
.6	.169 -.254	.138 -.249	.1 -.240	0 -.209	-.132 -.158	-.286 -.086	-.441 0	-.571 .086	-.658 .158	-.698 .209	-.700 .240	-.691 .249	-.678 .254
.4	.278 -.185	.254 -.183	.222 -.178	.132 -.158	0 -.121	-.167 -.067	-.345 0	-.400 .067	-.606 .121	-.658 .158	-.667 .178	-.660 .183	-.648 .185
.2	.392 -.098	.378 -.056	.357 -.095	.286 -.086	.167 -.067	0 -.037	-.192 0	-.370 .037	-.500 .067	-.571 .086	-.595 .095	-.595 .056	-.588 .098
0	.500 0	.497 0	.487 0	+.441 0	+.345 0	+.192 0	0 0	-.192 0	-.345 0	-.441 0	-.487 0	-.497 0	-.500 0

ρ_1 upper value in all, ρ_2 lower value in all.

Table 2

Values of $\{\phi_1, \rho_1, \rho_2, \rho_3\}$

ϕ_1	ρ_1	ρ_2	ρ_3
0	0	0	0
±.05	±.0501	.0025	±.0000
±.1	±.1010	.0102	±.0010
±.2	±.2087	.0436	±.0091
±.25	±.2679	.0718	±.0192
±.3	±.3333	.1111	±.0370
±.4	±.75	.5625	±.4291
±.5	±1	1	±1

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The general theory of stationary spatial models is developed: namely MA, moving average; AR, autoregressive; and ARMA, autoregressive moving average processes. As compared to the time series in m dimensions, spatial models may be one-sided, two-sided, or mixed. Free use is made of the previous results of Aroian and his associates in time series in m dimensions. The main theoretical properties of the models in the univariate case are established. The multivariate case is even more important than the univariate. Estimation by minimum variance and simulation of the models are included.

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